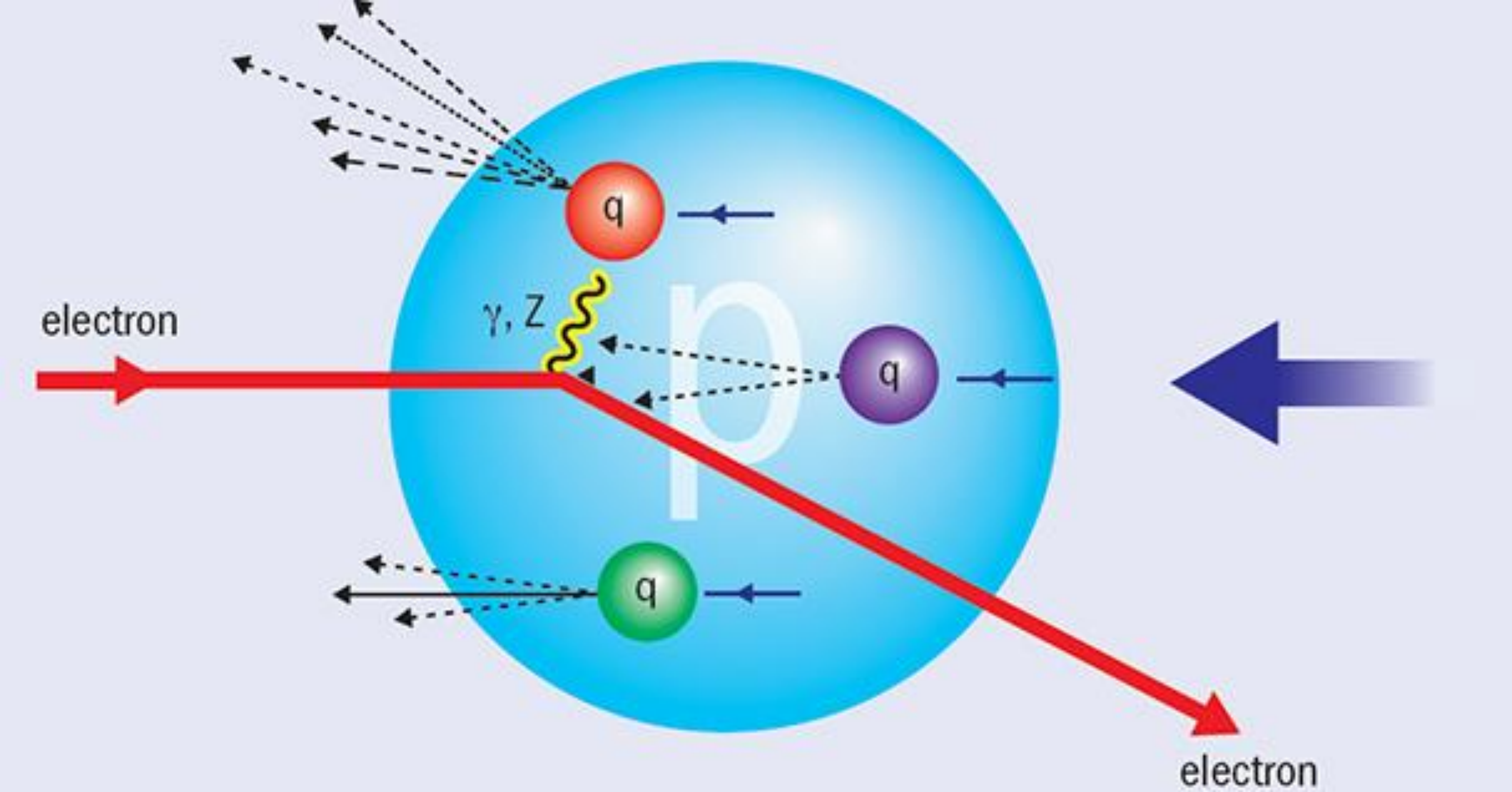

Particle Physics I

Lecture 9: Electron-proton elastic scattering

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Short recap and learning targets

- **Ultimate goal:** compute the cross section for electron-proton scattering, starting with the easier case of elastic scattering in various conditions

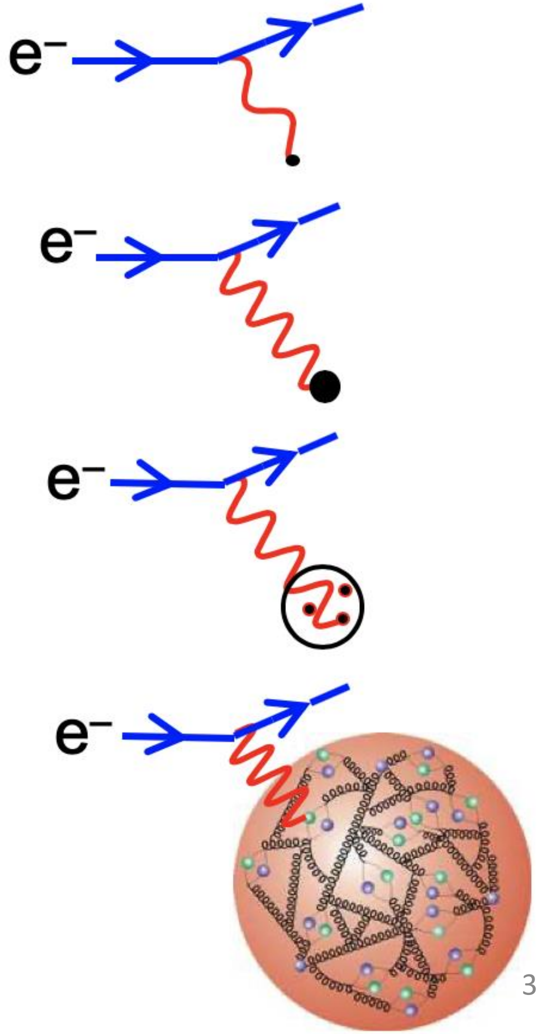
<u>Learning targets</u>

- Use the results from $e^+e^- \rightarrow \mu^+\mu^-$ to compute the matrix element for the QED part of *ep* scattering
- Scattering at low energy of the incoming electron ($E_e \ll m_e \ll M_p$): *Rutherford scattering*
- Scattering of relativistic electron with energy much smaller than the proton rest mass ($m_e \ll E_e \ll M_p$): *Mott scattering*
- Impact of the proton charge and magnetic moment distributions (Form Factors): scattering at even higher energies $\left(m_e \ll E_e \sim M_p\right)$
- How to measure the Form Factors – angular dependence and experimental considerations

Probing the structure of the proton

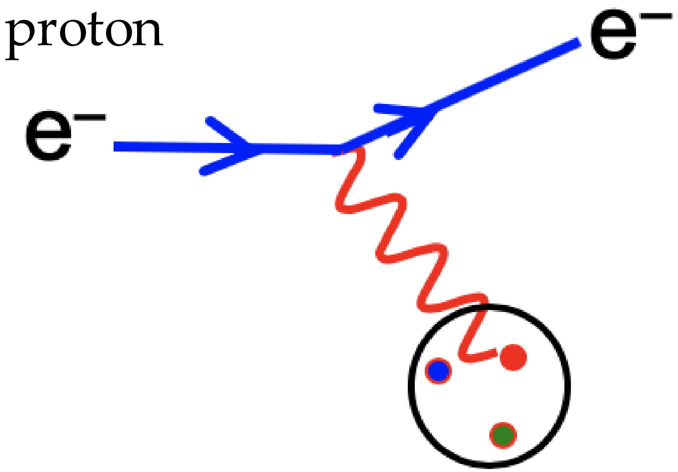
In $e^- p \to e^- p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on the photon wavelength

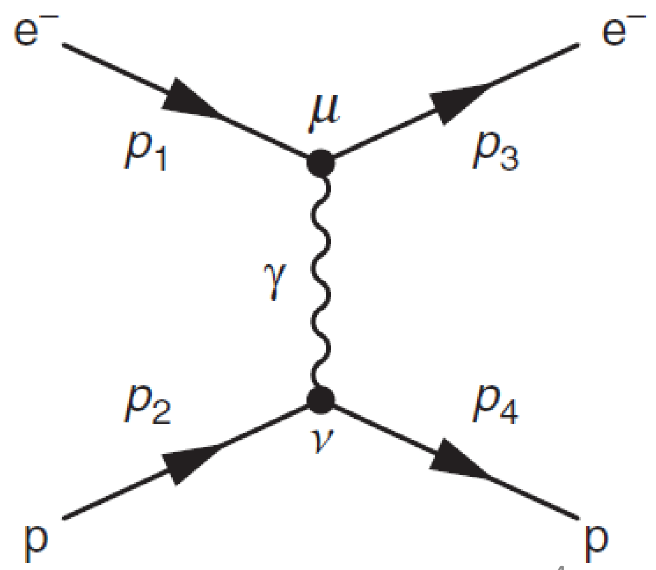
- At **very low** electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a “point-like” spin-less object
- At **low** electron energies $\lambda \sim r_p$: the scattering is equivalent to that from an extended charged object
- At **high** electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve the proton sub-structure. Scattering from constituent quarks
- At **very high** electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons



Electron-proton scattering

- Electron-proton scattering can be used as a probe of the structure of the proton
- Two main topics
  - $e^- p \to e^- p$: elastic scattering (today)
  - $e^- p \to e^- X$: deep inelastic scattering (next week)
- We will first consider scattering from a point-like proton





Electron-proton scattering

Two ways to proceed (derivations of the formulas below in Section 6.5.4)

1. Perform QED calculation from scratch:

$$\left\langle \left| M_{fi} \right|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)\right]$$

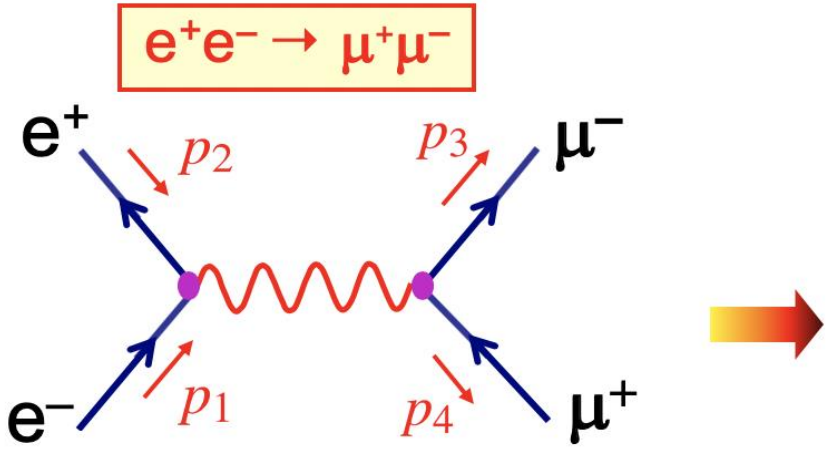
2. Take the results we derived from $e^+e^- \rightarrow \mu^+\mu^-$

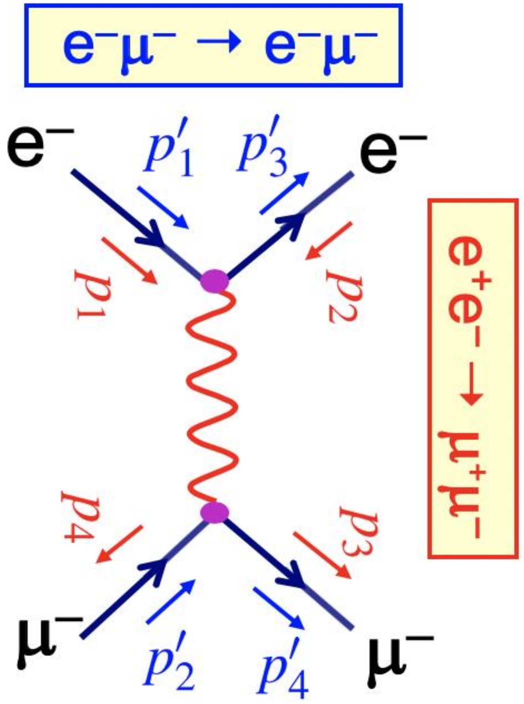
$$\left\langle \left| M_{fi} \right|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \frac{t^2 + u^2}{s^2}$$

and use "Crossing Symmetry" to obtain the matrix element for $e^-\mu^- \rightarrow e^-\mu^-$

Crossing symmetry

- We derived the Lorentz-invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ and we can now just "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element





$$p_1 \rightarrow p_1'; \; p_2 \rightarrow -p_3'; \; p_3 \rightarrow p_4'; \; p_4 \rightarrow -p_2'$$

Changes the spin-averaged matrix element for

$$\underset{p_1 \; p_2 \qquad p_3 \; p_4}{e^- e^+ \rightarrow \mu^- \mu^+} \;\Rightarrow\; \underset{p_1' \; p_2' \qquad p_3' \; p_4'}{e^- \mu^- \rightarrow e^- \mu^-}$$

Electron-proton scattering

- The calculated cross section is appropriate for scattering of two spin-half Dirac (i.e. point-like) particles in the ultra-relativistic limit ($E \gg m_{e,\mu}$) where we obtained

$e^+e^- \to \mu^+\mu^-$ $\qquad$ $e^-\mu^- \to e^-\mu^-$

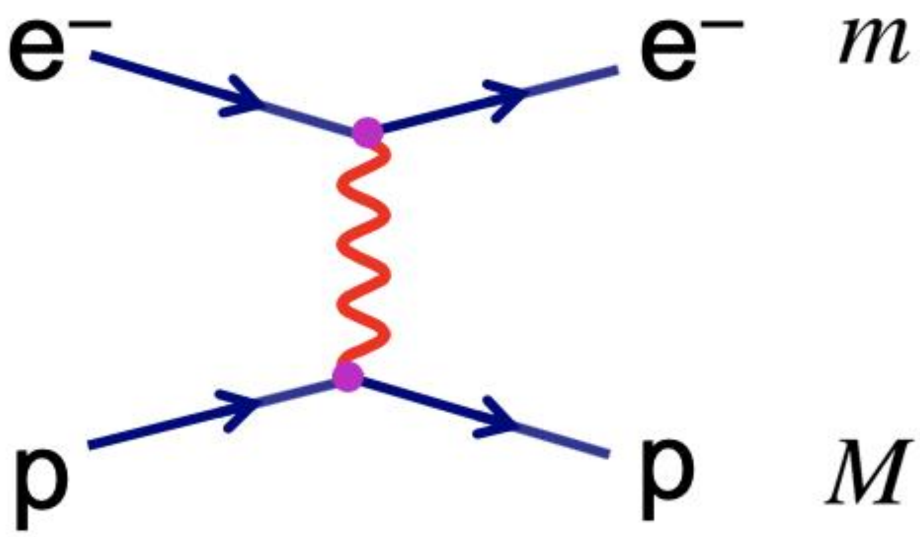
$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \frac{t^2 + u^2}{s^2} \Longrightarrow 2e^4 \frac{u^2 + s^2}{t^2}$$

$$p_1 \to p_1, p_2 \to -p_3$$
$$p_3 \to p_4, p_4 \to -p_2$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton next time

- Before doing so we will consider the scattering of electrons from the *composite* proton
  - how do we know that the proton is not a fundamental "point-like" particle?

Electron-proton scattering

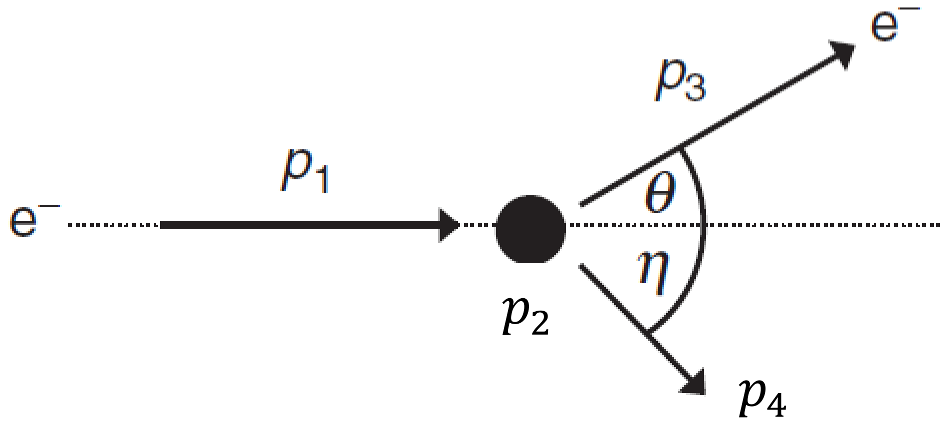


- In this discussion we will not be able to use the ultra-relativistic limit and will require the general expression for the matrix element (see Section 6.5.4):

$$\left\langle \left| M_{fi} \right|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M^2 - (p_1 \cdot p_4)m^2 + 2m^2M^2]$$

General $e^-p \to e^-p$ scattering allowing for proton recoil

- General description of scattering, where the proton recoil at an angle η is allowed
- Start from RH and LH helicity particle spinors



$$u_\uparrow = N\begin{pmatrix} c \\ se^{i\phi} \\ k\cdot c \\ k\cdot se^{i\phi}\end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ ce^{i\phi} \\ k\cdot s \\ -k\cdot ce^{i\phi}\end{pmatrix}$$

$$N = \sqrt{E+m},\ s = \sin(\theta/2),\ c = \cos(\theta/2),\ k = \frac{|\vec{p}|}{E+m} = \frac{\beta\gamma}{\gamma+1}$$

Non-relativistic limit: $k \to 0$

Ultra-relativistic limit: $k \to 1$

$$M_{fi} = \frac{e^2}{q^2}[\bar{u}(p_3)\gamma^\mu u(p_1)]g_{\mu\nu}[\bar{u}(p_4)\gamma^\nu u(p_2)] = \frac{e^2}{q^2} j_e \cdot j_p$$

Electron current $j^\mu_{(e)}$ Proton current $j^\nu_{(p)}$

Electron current

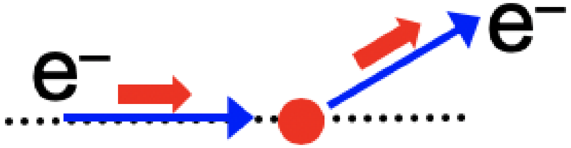
- The possible initial- and final-state electron spinors are:

initial − state electron: $\phi = 0, \theta = 0$ final − state electron: $\phi = 0, \theta$

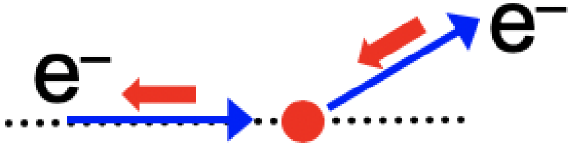
$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ k \\ 0 \end{pmatrix}, \quad u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -k \end{pmatrix}, \quad u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ kc \\ ks \end{pmatrix}, \quad u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ ks \\ -kc \end{pmatrix}$$

$$N_e = \sqrt{E + m_e}$$

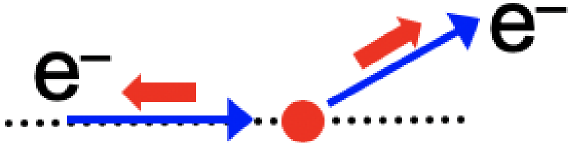
- Consider all four possible electron currents, i.e. helicities $R \to R, L \to L, L \to R, R \to L$:



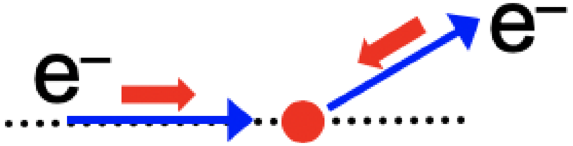
$$j^\mu_{e\uparrow\uparrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\big((k^2 + 1)c, 2ks, +2iks, 2kc\big)$$



$$j^\mu_{e\downarrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\big((k^2 + 1)c, 2ks, -2iks, 2kc\big)$$



$$j^\mu_{e\downarrow\uparrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\big((1 - k^2)s, 0, 0, 0\big)$$



$$j^\mu_{e\uparrow\downarrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\big((k^2 - 1)s, 0, 0, 0\big)$$

0 in the limit $k \to 1$

helicity=chirality

Proton current

- In the relativistic limit ($k = 1, E \gg m$) the currents $j_{e\uparrow\downarrow}$ and $j_{e\downarrow\uparrow}$ are 0 and only $R \to R$ and $L \to L$ currents contribute to the cross section

$$u_\uparrow(p_2 = 0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, u_\downarrow(p_2 = 0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, u_\uparrow(p_4) = \sqrt{2M_p}\begin{pmatrix}c_\eta\\-s_\eta\\0\\0\end{pmatrix}, u_\downarrow(p_4) = \sqrt{2M_p}\begin{pmatrix}-s_\eta\\-c_\eta\\0\\0\end{pmatrix},$$

- Giving for the proton currents

$$j^\nu_{p\uparrow\uparrow} = -j_{p\downarrow\downarrow} = +2M_p\left(c_\eta, 0,0,0\right)$$
$$j^\nu_{p\uparrow\downarrow} = +j_{p\downarrow\uparrow} = -2M_p\left(s_\eta, 0,0,0\right)$$

- The spin-averaged matrix element summing over all 8 helicity states is then

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} 4M_p^2(E + m_e)^2 \cdot \left[c_\eta^2 + s_\eta^2\right] \cdot [4(1 + k^2)^2 c^2 + 4(1 - k^2)^2 s^2]$$

Spin-averaged matrix element

Reworked equation from last slide (derive as an exercise)

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{4M_p^2 m_e^2 e^4 (\gamma_e + 1)^2}{q^4} \cdot \left[(1-k^2)^2 + 4k^2c^2\right]$$

We can use $k = \frac{\beta_e \gamma_e}{\gamma_e + 1}$ and $(1-\beta_e^2)\gamma_e^2 = 1$ to obtain

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{16 M_p^2 m_e^2 e^4}{q^4} \cdot \left[1 + \beta_e^2 \gamma_e^2 \cos^2\frac{\theta}{2}\right]$$

- Furthermore, in the t −channel: $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}$
- Giving us the general expression for elastic electron-proton scattering

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{M_p^2 m_e^2 e^4}{p^4 \sin^4\frac{\theta}{2}} \cdot \left[1 + \beta_e^2 \gamma_e^2 \cos^2\frac{\theta}{2}\right]$$

this term vanishes in the non-relativistic limit: $\beta_e \gamma_e \ll 1$

Rutherford scattering

- Rutherford scattering is the low-energy (non-relativistic) limit where the recoil of the proton can be neglected, and the electron is non-relativistic $(\beta_e \gamma_e \ll 1)$
- Using the expression for the differential cross section in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M_p + E_1(1-\cos\theta)}\right)^2 |M_{fi}|^2$$

- Here the electron is non-relativistic $E_1 \sim m_e \ll M_p$ and we can neglect E_1 in the denominator

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{e^4 m_e^2}{64\pi^2 |\vec{p}|^4 \sin^4\theta/2}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4\theta/2}$$

Rutherford scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4\theta/2}$$

- This is the normal expression for the Rutherford cross section
- We could derive it (and we did!) by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$ without any consideration of the interaction due to the intrinsic magnetic moments of the electron and proton
- Conclusion: in the non-relativistic limit, only the interaction between the particles' electric charges matter
- No contribution from the magnetic (spin-spin) interaction

Mott scattering cross section

- For Rutherford scattering we are in the limit where the target recoil is neglected, and the scattered particle is non-relativistic: $E_K \ll m_e$
- For Mott scattering the target recoil is neglected but the scattered particle is relativistic: $m_e \ll E \ll M_p$(i.e. we can neglect the electron mass)
  - in this limit ($E_1 = \gamma_e m_e, \beta_e \gamma_e \gg 1$) the matrix element becomes:

$$\langle |M_{fi}|^2 \rangle = \frac{M_p^2 e^4}{E_1^2 \sin^4 \frac{\theta}{2}} \cdot \cos^2 \frac{\theta}{2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \cdot \cos^2 \frac{\theta}{2}$$

Could have been derived from scattering of a relativistic electron off a spin-less nucleus

(we haven`t taken into account the charge distribution of the proton yet)

Mott scattering cross section

- The final result for the Mott scattering cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E_K^2 \sin^4\theta/2} \cdot \cos^2\frac{\theta}{2}$$

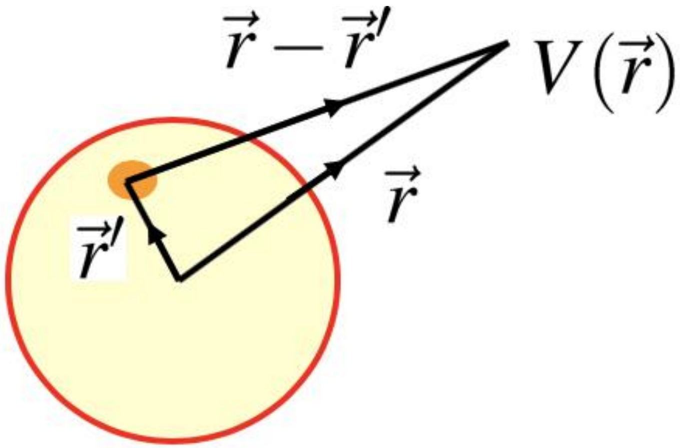
≈Rutherford formula with $E_K = E$

Overlap between initial- and final-state wave function just QM of spin-half

- *Note:* we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$. The interaction is electric rather than magnetic (spin-spin) in nature
- Last term arises from conservation of spin in the direction of motion for relativistic electrons (helicity/angular momentum conservation)
- We still haven`t taken into account the charge distribution of the proton

Form factors

- Consider the scattering of an electron in the static potential due to an extended charge distribution
- The potential at a distance $\vec{r}$ from the center is given by



$$V(\vec{r}) = \int \frac{Q\rho(\vec{r})}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) d^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by

$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3\cdot\vec{r}} V(\vec{r}) e^{+i\vec{p}_1\cdot\vec{r}} d^3\vec{r} = \int\int e^{i\vec{q}\cdot\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}'$$

$$= \int\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} e^{i\vec{q}\cdot\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}'$$

- Fix $\vec{r}'$ and integrate over $d^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

$$M_{fi} = \int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}\cdot\vec{r}'} d^3\vec{r}' = \left(M_{fi}\right)_{\text{point}} F(\vec{q}^2)$$

Form factors

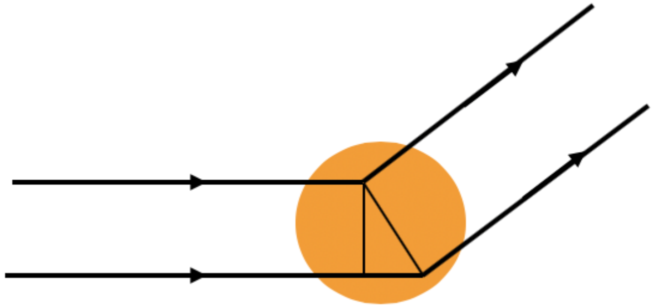
- The resulting matrix element is equivalent to the matrix element for scattering from a **point source** multiplied by the **form factor**

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

- We then get for the Mott scattering cross section

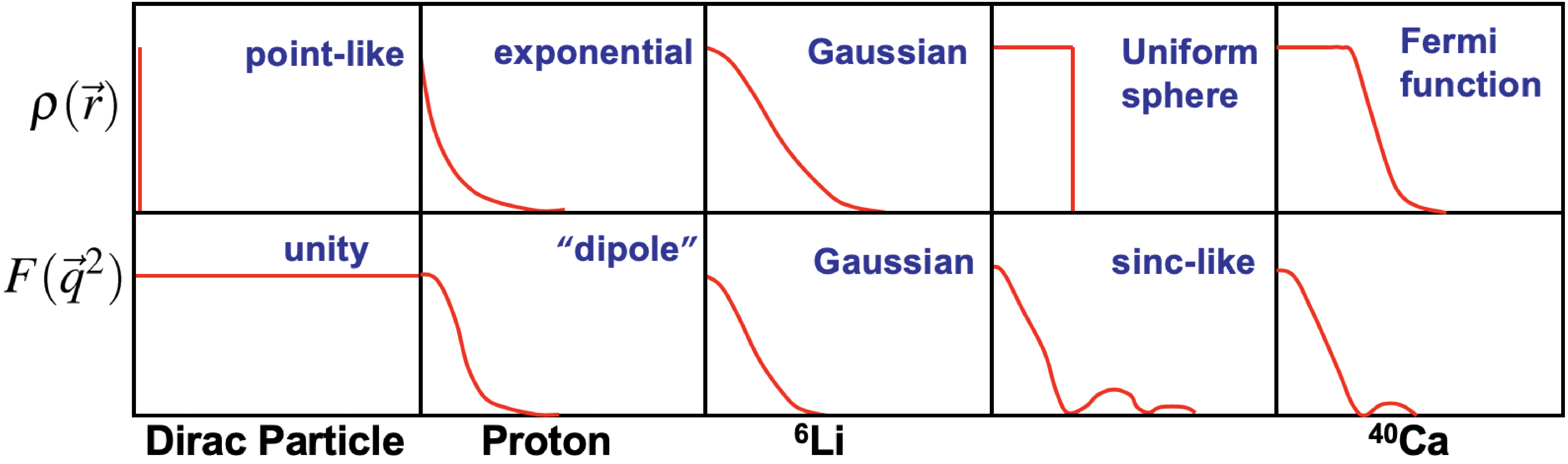
$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} \rightarrow \frac{\alpha^2}{4E_K^2 \sin^4\theta/2} \cdot \cos^2\frac{\theta}{2} \cdot |F(\vec{q}^2)|^2$$

- Form factors are similar to diffraction of plane waves in optics



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space"
- If the wavelength is long compared to the size of the object all waves are in phase and $F(\vec{q}^2) = 1$

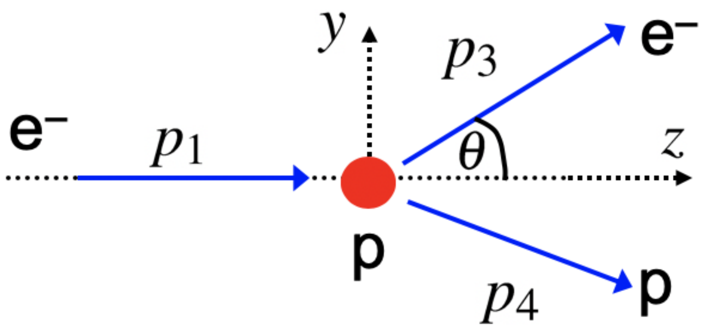
Form factors: examples



- *Note:* Form factor is unity for a point charge

Point-like Electron-proton ultra-relativistic elastic scattering

- So far, we have only considered protons which do not recoil. For $E_1 \gg m_e$ the general case is:



$$p_1 = (E_1, 0,0, E_1)$$

$$p_2 = \left(M_p, 0,0,0\right)$$

$$p_3 = (E_3, 0, E_3\sin\theta, E_3\cos\theta)$$

$$p_4 = (E_4, \vec{p}_4)$$

- Taking the equation from slide 8 with $m = m_e = 0$ the matrix element is

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M_p^2\right]$$

- Experimentally observe the scattered electron but not the proton so we need to eliminate p_4

- The scalar products not involving p_4 are

$$p_1 \cdot p_2 = E_1 M_p, \qquad p_1 \cdot p_3 = E_1 E_3 (1 - \cos\theta), \qquad p_2 \cdot p_3 = E_3 M_p$$

Point-like Electron-proton ultra-relativistic elastic scattering

- Using momentum conservation, we can eliminate p_4: $p_4 = p_1 + p_2 - p_3$

$$p_3 \cdot p_4 = p_3 \cdot p_1 + p_3 \cdot p_2 - p_3 \cdot p_3 \approx E_1 E_3 (1 - \cos\theta) + E_3 M_p$$

$$p_1 \cdot p_4 = p_1 \cdot p_1 + p_1 \cdot p_2 - p_1 \cdot p_3 \approx E_1 M_p - E_1 E_3 (1 - \cos\theta)$$

- Substituting the scalar products in the expression for the matrix element from the last slide we get

$$\left\langle \left| M_{fi} \right|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4} M_p E_1 E_3 \left[(E_1 - E_3)(1 - \cos\theta) + M_p (1 + \cos\theta) \right]$$

$$= \frac{8e^4}{(p_1 - p_3)^4} 2 M_p E_1 E_3 \left[(E_1 - E_3) \sin^2 \theta/2 + M_p \cos^2 \theta/2 \right]$$

- Now obtain expression for $q^4 = (p_1 - p_3)^4$ and $(E_1 - E_3)$

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2 p_1 \cdot p_3 = -2 E_1 E_3 (1 - \cos\theta) = -4 E_1 E_3 \sin^2 \theta/2 < 0 \text{ (space-like)}$$

Point-like Electron-proton ultra-relativistic elastic scattering

- For $(E_1 - E_3)$ we start from $q \cdot p_2 = (p_1 - p_3) \cdot p_2 = M(E_1 - E_3)$ and use

$$(q + p_2)^2 = p_4^2$$

$$q^2 + p_2^2 + 2q \cdot p_2 = p_4^2$$

$$q^2 + M_p^2 + 2q \cdot p_2 = M_p^2$$

$$\Longrightarrow q \cdot p_2 = -q^2/2$$

- Hence the energy transferred to the proton is

$$E_1 - E_3 = -\frac{q^2}{2M}$$

- Note: we found that $q^2 < 0$ and therefore $E_1 - E_3 > 0 \Longrightarrow$ the scattered electron always has lower energy than the incoming one

Interpretation

- So far, we derived the differential cross-section for $e^- p \to e^- p$ elastic scattering assuming point-like Dirac spin-half particles. How should we interpret the equation ?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\theta/2} \cdot \frac{E_3}{E_1} \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right)$$

- Compare with

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{16E_K^2 \sin^4\theta/2} \cdot \cos^2 \frac{\theta}{2}$$

Note: Mott cross section is equivalent to scattering of spin-half electrons in a fixed electro-static potential and the term E_3/E_1 is due to the proton recoil

- The new term $\propto \sin^2 \theta/2$: magnetic interaction due to spin-spin interaction

Interpretation

- The differential cross section from the previous slide **depends on a single parameter θ!**
- For an electron scattering at angle θ, both q^2 and E_3 are fixed by kinematics

$$-2M(E_1 - E_3) = -2E_1E_3(1 - \cos\theta)$$

$$\Rightarrow \frac{E_3}{E_1} = \frac{M}{M + E_1(1 - \cos\theta)}$$

- So, we can obtain for the exchanged momentum q^2

$$q^2 = -\frac{2ME_1^2(1 - \cos\theta)}{M + E_1(1 - \cos\theta)}$$

Interpretation

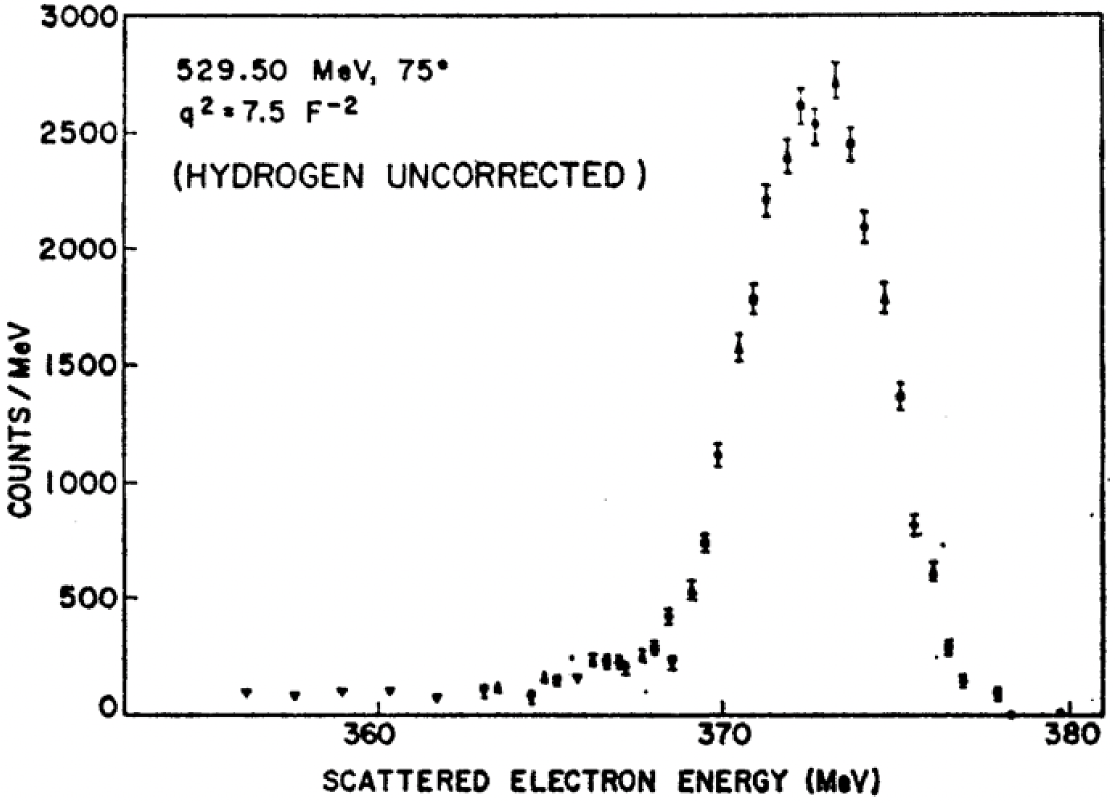
- Example: $e^- p \to e^- p$ at $E_{\text{beam}} = 529.5$ MeV, look at scattered electrons at $\theta = 75°$
- For **elastic** scattering we expect

$$E_3 = \frac{E_1 M}{M + E_1(1 - \cos\theta)}$$

$$E_3 = 938 \times \frac{529}{938 + 529(1 - \cos 75°)} = 373 \text{ MeV}$$

- The energy identifies the scattering as elastic
- We also know the squared four-momentum transfer

E.B. Hughes et.al., Phys. Rev. 139 (1965) B458



$$q^2 = -\frac{2 \times 938 \times 529^2(1 - \cos 75°)}{938 + 529(1 - \cos 75°)} = 0.294 \text{ GeV}^2$$

Elastic scattering from a finite-size proton

- In general, the finite size of the proton can be accounted for by introducing two structure functions
  - $G_E(q^2)$: related to the charge distribution inside the proton
  - $G_M(q^2)$: related to the distribution of the magnetic moment of the proton
- One can show that the differential cross section generalizes to the Rosenbluth formula:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \cdot \frac{E_3}{E_1}\left(\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right)$$

with the Lorentz-invariant quantity $\tau = -q^2/4M^2 > 0$

Elastic scattering from a finite-size proton

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the Fourier Transformation of the charge and magnetic moment distributions
- But we can relate them via $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and obtain

$$-\vec{q}^2 = q^2\left[1 - \left(\frac{q}{2M}\right)^2\right]$$

- So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

Elastic scattering from a finite-size proton

- In the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transformations of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$
$$G_M(q^2) \approx G_M(\vec{q}^2) = \int \mu(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

- *Note:* in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, so we expect for the magnetic moment of the proton

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

Elastic scattering from a finite-size proton

- But the experimentally measured value of the proton magnetic moment was found to be larger than what we expect for a point-like Dirac particle

$$\vec{\mu} = 2.79 \frac{e}{M} \vec{S}$$

- So for the **proton** we expect:

$$G_E(0) = \int \rho(\vec{r}) d^3\vec{r} = 1$$
$$G_M(0) = \int \mu(\vec{r}) d^3\vec{r} = \mu_p = +2.79$$

- The found anomalous magnetic moment of the proton is already evidence that it is not point-like!

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right), \text{where} \left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \cdot \frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

the Mott cross-section including the proton recoil corresponds to scattering from a spin-0 proton

- **At very low q^2:** $\tau = -\frac{q^2}{4M^2} \approx 0$

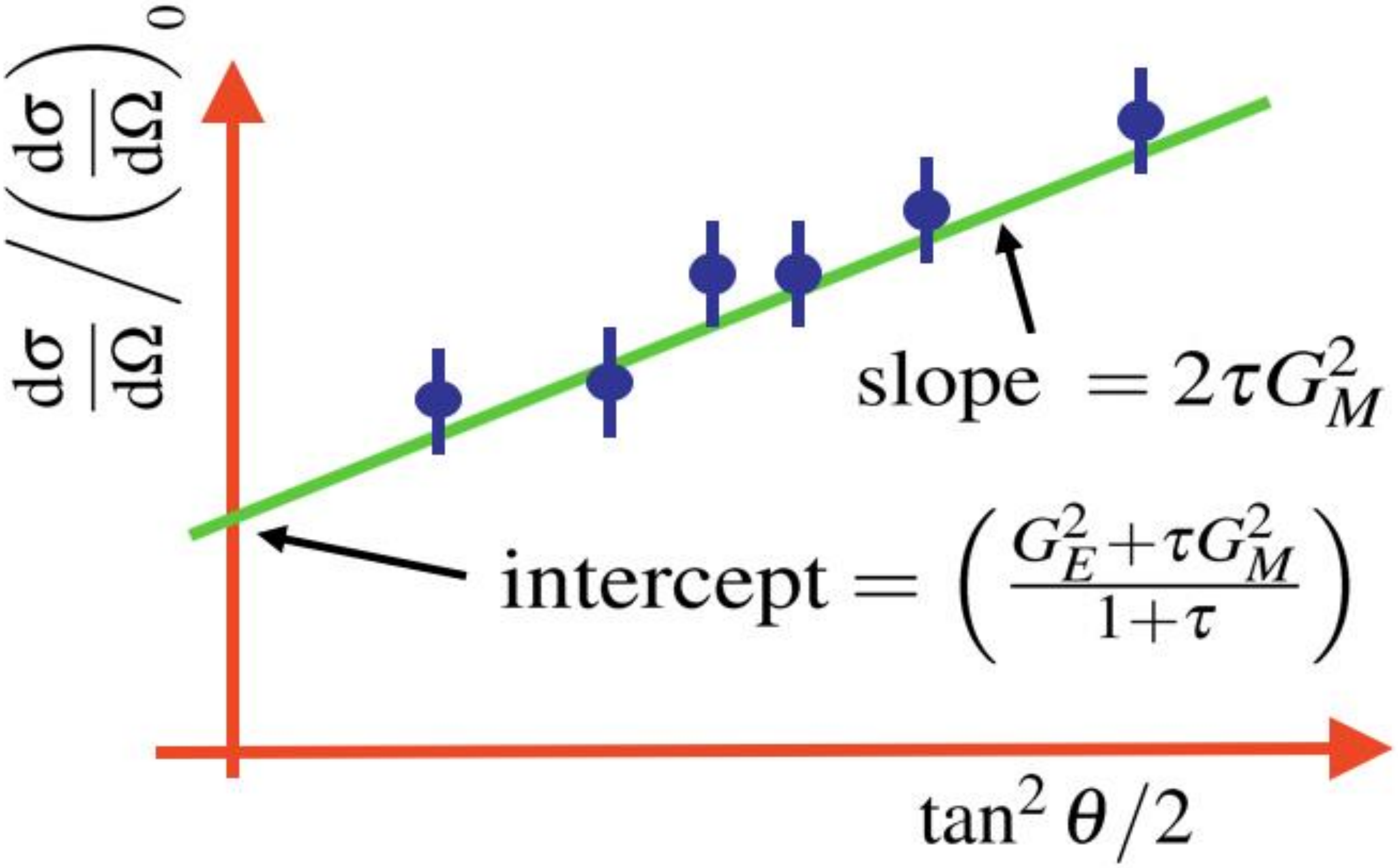
$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- **At high q^2:** $\tau \gg 1$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

Measuring $G_E(q^2)$ and $G_M(q^2)$

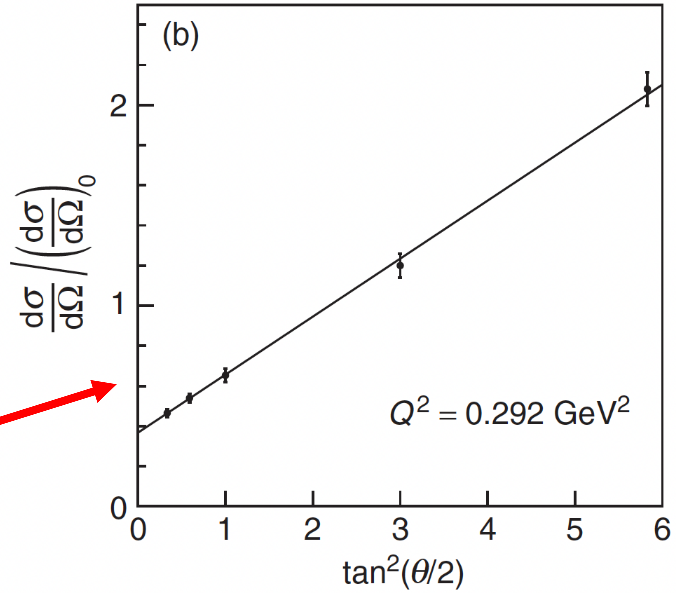
- In general, we are sensitive to both structure functions, which can be resolved from the angular dependence of the cross section at fixed q^2!



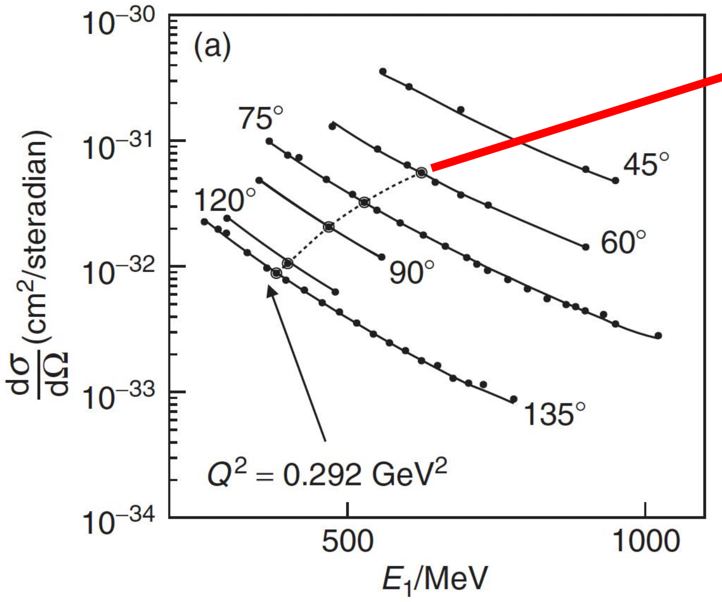
Measuring $G_E(q^2)$ and $G_M(q^2)$

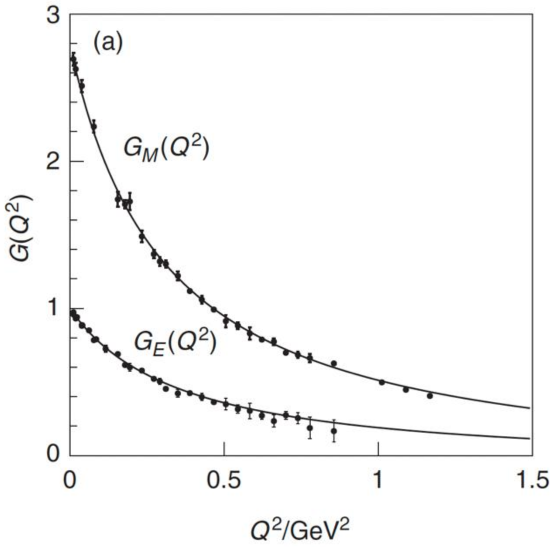
- Example: $e^- p \rightarrow e^- p$ at $E_{\text{beam}} = 529.5$ MeV
  - electron beam energies chosen to give certain values of q^2
  - cross section measured to $2 - 3\%$

E.B. Hughes et.al., Phys. Rev. 139 (1965) B458



$Q^2 = 0.292\text{GeV}^2$





Experimentally we find:

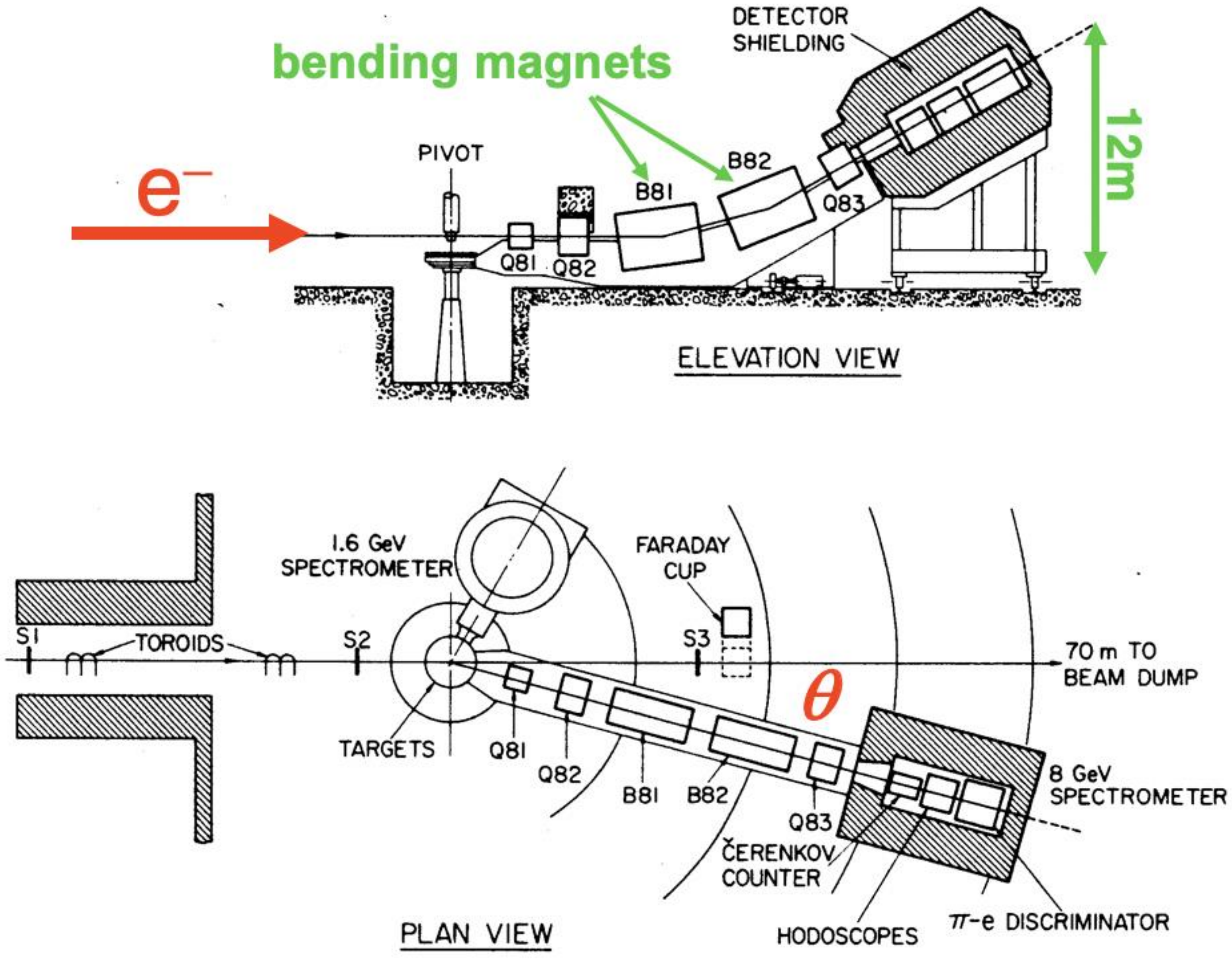
$$G_M(q^2) = 2.79 G_E(q^2)$$

the electric and magnetic form factors have the same distribution!

Higher energy electron-proton scattering

- Use electron beam the SLAC LINAC: $5 < E_{\text{beam}} < 20$ GeV

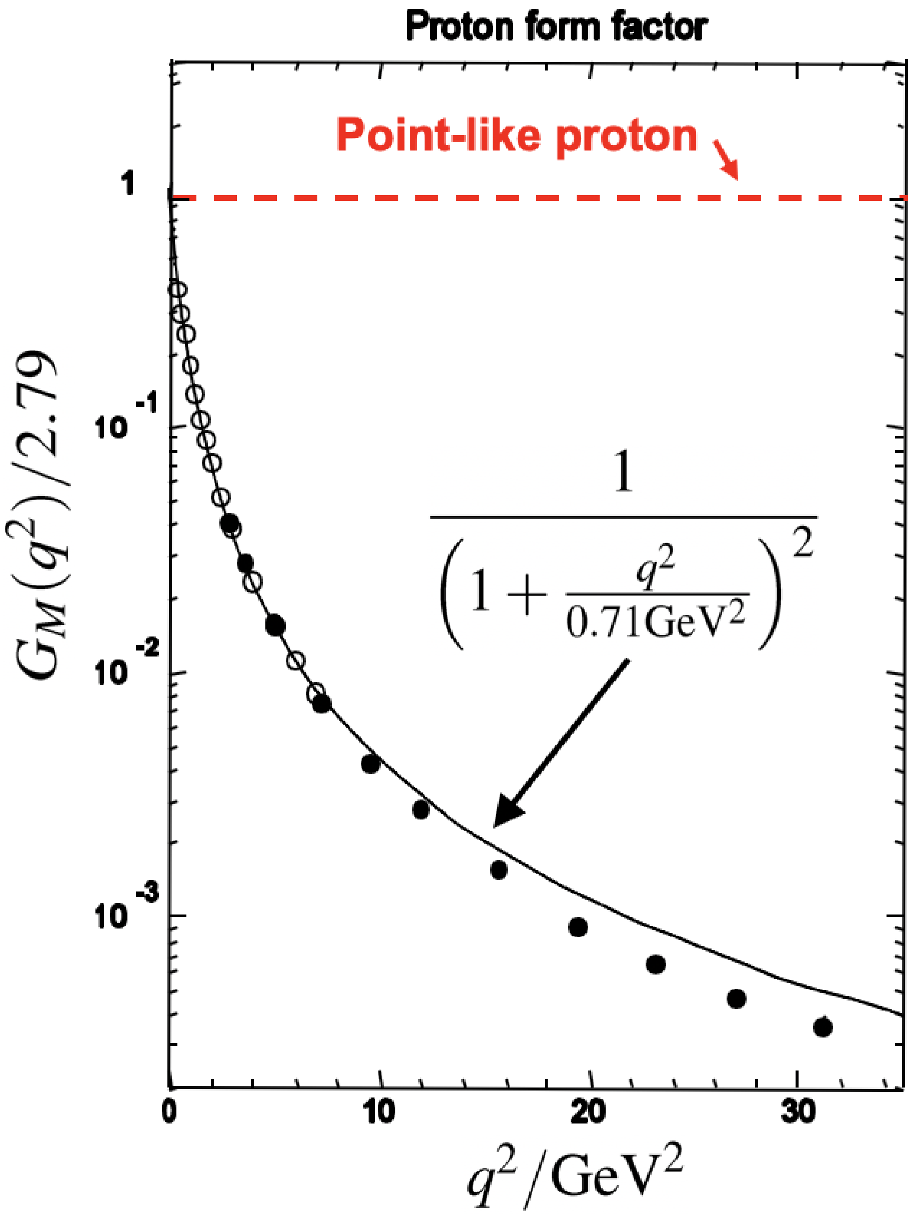
•Detect scattered electrons using the "8 GeV Spectrometer"



High q² → **Measure** $G_M(q^2)$

P.N.Kirk et.al., Phys. Rev. D 8 (1973) 63

High q^2 results



A.F. Sill *et.al.*, Phys. Rev. D 48 (1993) 29
R.C. Walker *et.al.*, Phys. Rev. D 49 (1994) 5671

- The form factor fall rapidly with q^2
  - proton is not point-like
  - decent fit (for low q^2) to the data with "dipole form"

$$G_M^p(q^2) = \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\text{GeV}^2)^2}$$

- Taking Fourier transformation, we find the spatial charge and magnetic moment distribution ($a \approx 0.24\text{fm}$)

$$\rho(r) \approx \rho_0 e^{-r/a}$$

  - corresponds to a rms charge radius $r_{rms} \approx 0.8$ fm
- Although suggestive, does not imply proton is composite!
- *Note:* so far, we have only considered elastic scattering (inelastic scattering next week!

Summary of Lecture 9

<u>Main learning outcomes</u>

- How to obtain the cross section for elastic *ep* scattering starting
  - starting from the QED process $e^+e^- \rightarrow \mu^+\mu^-$ and relating it to the QED part of *ep* scattering
  - low-energy *ep* scattering: *Rutherford*
  - high-energy *ep* scattering (no proton recoil): *Mott*
  - considering proton recoil, spin-spin interaction, electric and magnetic Form Factors